High School Geometry: Instrument of the Devil

There is nothing quite so vexing to the author of a scathing indictment as having the primary target of his venom offered up in his support. And never was a wolf in sheep's clothing as insidious, nor a false friend as treacherous, as High School Geometry. It is precisely *because* it is school's attempt to introduce students to the art of argument that makes it so very dangerous.

Posing as the arena in which students will finally get to engage in true mathematical reasoning, this virus attacks mathematics at its heart, destroying the very essence of creative rational argument, poisoning the students' enjoyment of this fascinating and beautiful subject, and permanently disabling them from thinking about math in a natural and intuitive way.

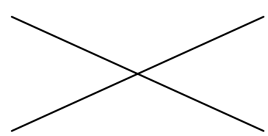
The mechanism behind this is subtle and devious. The student-victim is first stunned and paralyzed by an onslaught of pointless definitions, propositions, and notations, and is then slowly and painstakingly weaned away from any natural curiosity or intuition about shapes and their patterns by a systematic indoctrination into the stilted language and artificial format of so-called "formal geometric proof."

All metaphor aside, geometry class is by far the most mentally and emotionally destructive component of the entire K-12 mathematics curriculum. Other math courses may hide the beautiful bird, or put it in a cage, but in geometry class it is openly and cruelly tortured. (Apparently I am incapable of putting all metaphor aside.)

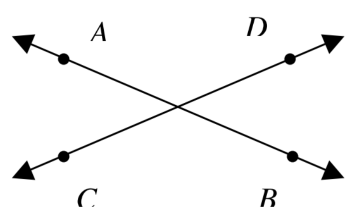
What is happening is the systematic undermining of the student's intuition. A proof, that is, a mathematical argument, is a work of fiction, a poem. Its goal is to *satisfy*. A beautiful proof should explain, and it should explain clearly, deeply, and elegantly. A well-written, well-crafted argument should feel like a splash of cool water, and be a beacon of light— it should refresh the spirit and illuminate the mind. And it should be *charming*.

There is nothing charming about what passes for proof in geometry class. Students are presented a rigid and dogmatic format in which their so-called "proofs" are to be conducted— a format as unnecessary and inappropriate as insisting that children who wish to plant a garden refer to their flowers by genus and species.

Let's look at some specific instances of this insanity. We'll begin with the example of two crossed lines:



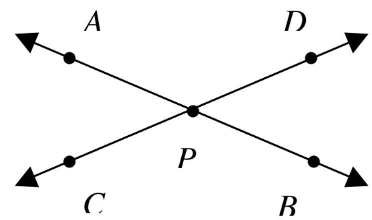
Now the first thing that usually happens is the unnecessary muddying of the waters with excessive notation. Apparently, one cannot simply speak of two crossed lines; one must give elaborate names to them. And not simple names like 'line 1' and 'line 2,' or even '*a*' and '*b*.' We must (according to High School Geometry) select random and irrelevant points on these lines, and then refer to the lines using the special "line notation."



You see, now we get to call them *AB* and *CD*. And God forbid you should omit the little bars on top— '*AB*' refers to the *length* of the line *AB* (at least I think that's how it works). Never mind how pointlessly complicated it is, this is the way one must learn to do it. Now comes the actual statement, usually referred to by some absurd name like

PROPOSITION 2.1.1.

Let $\overline{AB}$ and $\overline{CD}$ intersect at P. Then $\angle APC \cong \angle BPD$.



In other words, the angles on both sides are the same. Well, duh! The configuration of two crossed lines is *symmetrical* for crissake. And as if this wasn't bad enough, this patently obvious statement about lines and angles must then be "proved."

Proof:

Statement	Reason
1. $m\angle APC + m\angle APD = 180$ $m\angle BPD + m\angle APD = 180$	1. Angle Addition Postulate
2. $m\angle APC + m\angle APD = m\angle BPD + m\angle APD$	2. Substitution Property
3. $m\angle APD = m\angle APD$	3. Reflexive Property of Equality
4. $m\angle APC = m\angle BPD$	4. Subtraction Property of Equality
5. $\angle APC \cong \angle BPD$	5. Angle Measurement Postulate

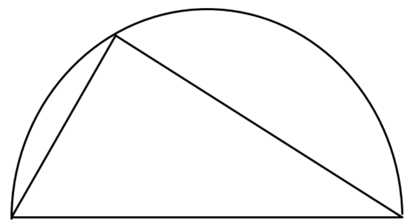
Instead of a witty and enjoyable argument written by an actual human being, and conducted in one of the world's many natural languages, we get this sullen, soulless, bureaucratic form-letter of a proof. And what a mountain being made of a molehill! Do we really want to suggest that a straightforward observation like this requires such an extensive preamble? Be honest: did you actually even read it? Of course not. Who would want to?

The effect of such a production being made over something so simple is to make people doubt their own intuition. Calling into question the obvious, by insisting that it be "rigorously proved" (as if the above even constitutes a legitimate formal proof) is to say to a student, "Your feelings and ideas are suspect. You need to think and speak our way."

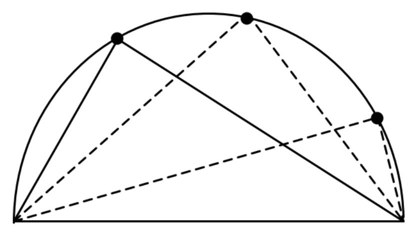
Now there is a place for formal proof in mathematics, no question. But that place is not a student's first introduction to mathematical argument. At least let people get familiar with some mathematical objects, and learn what to expect from them, before you start formalizing everything. Rigorous formal proof only becomes important when there is a *crisis*— when you discover that your imaginary objects behave in a counterintuitive way; when there is a paradox of some kind. But such excessive preventative hygiene is completely unnecessary here— nobody's gotten sick yet! Of course if a logical crisis should arise at some point, then obviously it should be investigated, and the argument made more clear, but that process can be carried out intuitively and informally as well. In fact it is the soul of mathematics to carry out such a dialogue with one's own proof.

So not only are most kids utterly confused by this pedantry— nothing is more mystifying than a proof of the obvious— but even those few whose intuition remains intact must then retranslate their excellent, beautiful ideas back into this absurd hieroglyphic framework in order for their teacher to call it "correct." The teacher then flatters himself that he is somehow sharpening his students' minds.

As a more serious example, let's take the case of a triangle inside a semicircle:

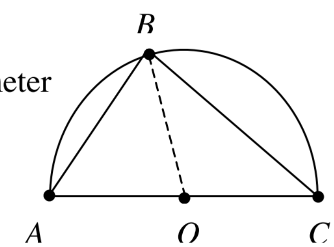


Now the beautiful truth about this pattern is that no matter where on the circle you place the tip of the triangle, it always forms a nice right angle. (I have no objection to a term like "right angle" if it is relevant to the problem and makes it easier to discuss. It's not terminology itself that I object to, it's pointless unnecessary terminology. In any case, I would be happy to use "corner" or even "pigpen" if a student preferred.)



Here is a case where our intuition is somewhat in doubt. It's not at all clear that this should be true; it even seems *unlikely*— shouldn't the angle change if I move the tip? What we have here is a fantastic math problem! Is it true? If so, *why* is it true? What a great project! What a terrific opportunity to exercise one's ingenuity and imagination! Of course no such opportunity is given to the students, whose curiosity and interest is immediately deflated by:

THEOREM 9.5. Let ΔABC be inscribed in a semicircle with diameter $\overline{A}\ \overline{C}$.
Then $\angle ABC$ is a right angle.



Proof:

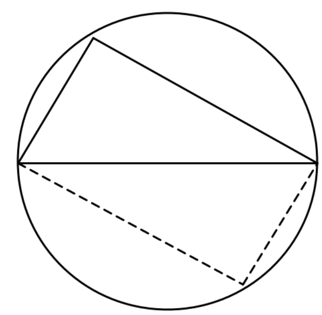
Statement	Reason
1. Draw radius *OB*. Then $OB = OC = OA$	1. Given
2. $m\angle OBC = m\angle BCA$ $m\angle OBA = m\angle BAC$	2. Isosceles Triangle Theorem
3. $m\angle ABC = m\angle OBA + m\angle OBC$	3. Angle Sum Postulate
4. $m\angle ABC + m\angle BCA + m\angle BAC = 180$	4. The sum of the angles of a triangle is 180
5. $m\angle ABC + m\angle OBC + m\angle OBA = 180$	5. Substitution (line 2)
6. $2\,m\angle ABC = 180$	6. Substitution (line 3)
7. $m\angle ABC = 90$	7. Division Property of Equality
8. $\angle ABC$ is a right angle	8. Definition of Right Angle

Could anything be more unattractive and inelegant? Could any argument be more obfuscatory and unreadable? This isn't mathematics! A proof should be an epiphany from the Gods, not a coded message from the Pentagon. This is what comes from a misplaced sense of logical rigor: *ugliness*. The spirit of the argument has been buried under a heap of confusing formalism.

No mathematician works this way. No mathematician has *ever* worked this way. This is a complete and utter misunderstanding of the mathematical enterprise. Mathematics is not about erecting barriers between ourselves and our intuition, and making simple things complicated. Mathematics is about removing obstacles to our intuition, and keeping simple things simple.

Compare this unappetizing mess of a proof with the following argument devised by one of my seventh-graders:

"Take the triangle and rotate it around so it makes a four-sided box inside the circle. Since the triangle got turned completely around, the sides of the box must be parallel, so it makes a parallelogram. But it can't be a slanted box because both of its diagonals are diameters of the circle, so they're equal, which means it must be an actual rectangle. That's why the corner is always a right angle."



Isn't that just delightful? And the point isn't whether this argument is any better than the other one *as an idea*, the point is that the idea comes across. (As a matter of fact, the idea of the first proof is quite pretty, albeit seen as through a glass, darkly.)

More importantly, the idea was the student's *own*. The class had a nice problem to work on, conjectures were made, proofs were attempted, and this is what one student came up with. Of course it took several days, and was the end result of a long sequence of failures.

To be fair, I did paraphrase the proof considerably. The original was quite a bit more convoluted, and contained a lot of unnecessary verbiage (as well as spelling and grammatical errors). But I think I got the feeling of it across. And these defects were all to the good; they gave me something to do as a teacher. I was able to point out several stylistic and logical problems, and the student was then able to improve the argument. For instance, I wasn't completely happy with the bit about both diagonals being diameters— I didn't think that was entirely obvious— but that only meant there was more to think about and more understanding to be gained from the situation. And in fact the student was able to fill in this gap quite nicely:

> "Since the triangle got rotated halfway around the circle, the tip must end up exactly opposite from where it started. That's why the diagonal of the box is a diameter."

So a great project and a beautiful piece of mathematics. I'm not sure who was more proud, the student or myself. This is exactly the kind of experience I want my students to have.